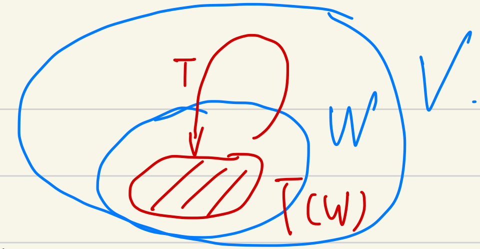
§ Invariant Subspace



Def: $T \in \mathcal{L}(V)$. A subspace $W \subset V$ is T-invariant if $T(W) \subset W$, i.e., $T(\vec{w}) \in W$, $\forall \vec{w} \in W$

Example: (1) $T \in \mathcal{L}(V)$, the following subspaces of V are T-invariant

- $\{0\}$.
- V
- $R(T)$ $\quad (w \in R(T) \subset V,\ T(\vec{w}) \in R(T))$
- $N(T)$
- $E_\lambda = N(T - \lambda I)$ eigenspace $\quad$ (if $\vec{w} \in E_\lambda$. $T(\vec{w}) = \lambda \cdot \vec{w} \in E_\lambda$)

In fact 1-dim T-inv. space $\{c \cdot \vec{w}\}$ $\Leftrightarrow$ $\vec{w}$ eigenvalue.

(2) $T = L_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

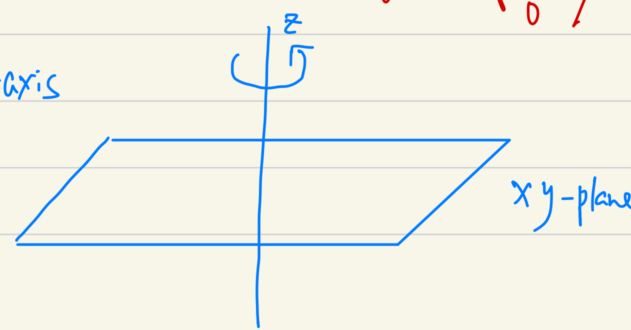
x-axis. $\{(x,0,0) : x \in \mathbb{R}\}$ is T-invariant.

Z-axis $\{(0,0,z) : z \in \mathbb{R}\}$ is T-invariant

y-axis. $\{(0,y,0) : y \in \mathbb{R}\}$ is NOT. T-invariant. $\left(T\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 3y \\ 0 \end{pmatrix} \right)$

xy-plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ is T-invariant $\left(T\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 3x+y \\ 3y \\ 0 \end{pmatrix} \right)$

(3). $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ rotation along z-axis



xy-plane, z-axis T-inv.

Def: $T \in \mathcal{L}(V)$. Given a non-zero $\vec{x} \in V$.

the subspace $W := \operatorname{span}(\{\underbrace{T^k}_{:= T \circ T \circ T \cdots \circ T}(\vec{x}) : k \in \mathbb{N}\}) = \operatorname{span}(\{\vec{x}, T(\vec{x}), T^2(\vec{x}), \cdots\})$ is called the **T-cyclic subspace of V generated by $\vec{x}$**.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.

Pf:

- For any $\vec{w} \in W$, suppose $\vec{w} = \sum_{i=0}^{k} a_i T^i(\vec{x})$

$$\Rightarrow \quad T(\vec{w}) = \sum_{i=0}^{k} a_i T^{i+1}(k) \in W.$$

Hence, W is T-invariant.

- If $U \subset V$ T-inv subspace containing $\vec{x}$, then it contains $T(\vec{x})$, $T(T(\vec{x})) = T^2(\vec{x})$, and more generally, $T^k(\vec{x})$ $\forall k$. Hence $W \subset U$.

$\square$

Example: (1) $T: P(\mathbb{R}) \longrightarrow P(\mathbb{R})$ defined by $T(f(x)) = f'(x)$.

the cyclic subspace gen. by x^n is

$$\mathrm{Span}(\{x^n, nx^{n-1}, n(n-1)x^{n-2}, \cdots, n!\cdot x, n!\}) = P_n(\mathbb{R}).$$

(2) $T = L_A: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. $\quad A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

- cyclic subspace gen. by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is x-axis.

(More generally, $\vec{v} \in V$ is an eigenvector of T, T-inv. subspace gen. by $\vec{v}$ is $\mathrm{span}\{\vec{v}\}$)

- cyclic subspace gen. by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is xy-plane.

Def: A vector $\vec{x}$ is called generalized eigenvector of T corresponding to λ if $(T-\lambda I)^p \cdot \vec{x} = 0$. for some positive integer p.

Observe: $(T-\lambda I)\underbrace{\left((T-\lambda I)^{p-1}\vec{x}\right)}=0$. $\Rightarrow$ λ is eigenvalue of T

Def: Generalized eigenspace. $K_\lambda := \{\vec{x}\in V : (T-\lambda I)^p \cdot \vec{x} = 0 \text{ for some } p>0\}$

Prop: (1) $K_\lambda \supset E_\lambda$ where $E_\lambda := \{\vec{x}\in V.\ (T-\lambda I)\vec{x}=0\}$

(2). K_λ is a T-invariant subspace.

Example: $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\lambda_1 = 3$, $\lambda_2 = 4$.

Note, $(A-3I)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $(A-3I)^2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = (A-3I)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Hence $\begin{pmatrix}0\\1\\0\end{pmatrix} \in K_{\lambda=3}$.

In fact, $K_{\lambda=3}$ = xy-plane. $\supset E_{\lambda=3}$ = x-axis

$K_{\lambda=4} = E_{\lambda=4}$ = z-axis.

Suppose $T \in \mathcal{L}(V)$, W: T-invariant Subspace of V.

Restricted to W, obtain $T_W : W \longrightarrow W$ linear operator on W.

<u>Prop</u>: Characteristic poly. $f_{T_W}(t)$ divides $f_T(t)$

pf: Choose an ordered basis $\gamma = \{\vec{v_1}, \cdots, \vec{v_k}\}$ for W and extend it to an ordered basis $\beta = \{\vec{v_1}, \cdots, \vec{v_k}, \vec{v_{k+1}}, \cdots, \vec{v_n}\}$ for V.

Then, $$[T]_\beta = \begin{pmatrix} | & | & \\ \underbrace{[T(\vec{v_1})]_\beta}_{\in W} & \underbrace{[T(\vec{v_2})]_\beta}_{\in W} & \cdots \\ | & | & \end{pmatrix} = \begin{array}{c|c|c} & k & n-k \\ k & [T_W]_\gamma & B \\ n-k & O & C \end{array}$$

$$= \begin{array}{c} k \\ n-k \end{array} \begin{pmatrix} [T(\vec{v_1})]_\gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence, $f_T(t) = \det\left([T]_\beta - t\cdot I\right)$

$$= \det \begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \left(\begin{array}{c|c} [T_W]_\gamma - tI_k & B \\ \hline O & C - t\cdot I_{n-k} \end{array}\right) \end{array}$$

$$= \det\left([T_W]_\gamma - tI_k\right) \cdot \underbrace{\det\left(C - tI_{n-k}\right)}_{g(t).}$$

$$= f_{T_W}(t)\cdot g(t)$$

So $f_{T_W}(t)$ divides $f_T(t)$. $\square$

Theorem: $T \in \mathcal{L}(V)$, V finite-dim. $W = \operatorname{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \cdots\}$

Let $W \subset V$ be the T-cyclic Subspace gen. by $\vec{v} \in V$.

Assume $\dim W = k$

Then. (a). $\{\vec{v}, T(\vec{v}), \cdots, T^{k-1}(\vec{v})\}$ is a basis for W.

(b). If $\boxed{a_0 \vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = 0.}$

then the characteristic polynomial of $T_W \in \mathcal{L}(W)$

is. $\boxed{f_{T_W}(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)}$.

Pf: (a). Suppose j is the largest integer s.t. $\beta = \{\vec{v}, T(\vec{v}), \cdots, T^{j-1}(\vec{v})\}$ is lin. indep.

Such j exists because V is finite dimension.

$$Z := \operatorname{span}(\beta).$$

- Z is T-invariant:

Note $T^j(\vec{v}) \in Z$ since $\beta \cup \{T^j(\vec{v})\}$ is lin. dependent by assumption

For $\vec{w} \in Z$, write $\vec{w} = b_0 \vec{v} + b_1 T(\vec{v}) + \cdots + b_{j-1} T^{j-1}(\vec{v})$.

then $T(\vec{w}) = b_0 T(\vec{v}) + b_1 T^2(\vec{v}) + \cdots + b_{j-1} T^j(\vec{v}) \in Z$.

$\Rightarrow$ Z is T-invariant.

- Since W is the smallest T-inv. subspace containing $\vec{v}$.

  We have $W \subset Z$.

$$\operatorname{span}\{\vec{v}, T(\vec{v}), \dots, T^{j-1}(\vec{v})\}$$
$$\parallel$$

But by definition, we also have $Z \subset W = \operatorname{span}\{\vec{v}, T(\vec{v}), T(v), \dots\}$

Hence $W = Z$, and β is a basis for W.

Thus $j = k$. and this proves (a).

Part (b):

$$\beta = \{ \underbrace{\vec{v}}_{=\vec{v}_1}, \underbrace{T(\vec{v})}_{=\vec{v}_2}, \ldots, \underbrace{T^{k-1}(\vec{v})}_{=\vec{v}_k} \}$$ ordered basis for W.

then $$[T_W]_\beta = \begin{pmatrix} | & | & & | \\ [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_k)]_\beta \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ [T(\vec{v})]_\beta & [T^2(\vec{v})]_\beta & \cdots & [T^k(\vec{v})]_\beta \\ | & | & & | \end{pmatrix}$$

$T^k(\vec{v}) = -a_0\vec{v} - a_1 T(\vec{v}) \cdots - a_{k-1} T^{k-1}(\vec{v})$

$$= \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & \cdots & \vdots & \vdots \\ 0 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

$$f_{T_W}(t) = \det([T_W]_\beta - tI) = \det \begin{pmatrix} -t & & & & -a_0 \\ 1 & -t & & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & -t & \vdots \\ \vdots & \vdots & & & -a_{k-2} \\ 0 & 0 & & 1 & -a_{k-1}-t \end{pmatrix}$$

$$= (-1)^k \cdot (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$$

(HW)

$\square$

Theorem (Cayley-Hamilton)

Let $T \in \mathcal{L}(V)$ with $\dim V = n$. and let $f(t) = f_T(t)$ char. poly.

Then $f(T) =$ Zero transformation on V.

More explicitly, if $f(t) = a_0 1 + a_1 t + \cdots + a_n t^n$

then $f(T) = a_0 I + a_1 T + \cdots + a_n T^n = 0 \in \mathcal{L}(V)$

Pf: We want to show $f(T)(\vec{v}) = 0$ for all $\vec{v} \in V$.

Let $W = T$-cyclic subspace gen. by $\vec{v}$, $\dim W = k$

By Thm. above, if $g(t) := f_{T_W}(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$

then $g(T) \cdot (\vec{v}) = 0$.

Recall $g(t) = f_{T_W}(t)$ divides $f(t) = f_T(t)$

$\Rightarrow f(t) = q(t) \cdot g(t)$

Thus, $f(T)(\vec{v}) = q(T) \cdot \underbrace{g(T) \cdot (\vec{v})}_{=0} = 0$. $\square$

Corollary: Let $A \in M_{n\times n}(F)$ and $f(t)$ be its char. poly.

Then $f(A) = O$ the zero Matrix.

- Application of Cayley-Hamilton:

1) Inverse matrix:
$$f_A(t) = (-1)^n \cdot t^n + \overset{(-1)^{n-1}\cdot \operatorname{tr}A}{a_{n-1}}\, t^{n-1} + \cdots + a_1 t + \overset{\det A}{a_0}$$

$$\text{Cayley-Hamilton} \Rightarrow (-1)^n \cdot A^n + a_{n-1}\cdot A^{n-1} + \cdots + a_1 A + \det A \cdot I_n = 0.$$

$$\Rightarrow \boxed{A^{-1} = -\frac{1}{\det A}\cdot\left((-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 \cdot I_n\right)}$$

In particular, if A is 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $A^{-1} = -\frac{1}{\det A} \cdot (A - \operatorname{tr}A \cdot I) = \frac{1}{\det A} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

2) n^{th} power of matrix; e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad f_A(t) = t^2 - 5t - 2$

Cayley-Hamilton $\Rightarrow$ $A^2 = 5A + 2 \cdot I$

$A^3 = 5A^2 + 2A = 27A + 10I \quad \dots$